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The singularity in axially symmetric fields

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Abstract. A static axially symmetric solution of the Einstein–Maxwell-scalar equations is obtained. For vanishing electric and scalar charges the metric reduces to the Curzon metric. The effect of these fields in the structure of the singularities has been analysed.

1. Introduction

In a paper, Gautreau and Anderson (1967) evaluated the Kretschmann scalar $R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma}$ for an axially symmetric Weyl gravitational field. They investigated the behaviour of this invariant in the case of the Curzon (1924) metric. From the analysis of this scalar it is found that the singular behaviour of the scalar depends on the direction in which the singular point is approached. Later, Stachel (1968) demonstrated that the Curzon metric contains a singular event horizon of infinite area for positive mass. The approach to the origin along different directions corresponds to an approach to different limiting points on the infinite surface.

In a subsequent paper, Gautreau (1969) studied several Weyl fields to determine the changes that occur in the topological properties of the singularities with the coupling of a long-range scalar field. Later, Datta and Rao (1973) extended the study of Gautreau to the case when the electromagnetic field is also coupled. They found that their solution did not show any directional singularity. However, in computing the Kretschmann scalar, they did not take into account the contribution from an exponential factor multiplying all the terms of the scalar. The contribution of the exponential factor is contrary to their conclusion.

In the present paper, we re-examine the topological properties of the singularities of a static, axially symmetric metric corresponding to the coupled gravitational, electromagnetic and long-range scalar fields. We obtain a more general class of exact solution than those already known for the coupled field equations with the Curzon solution as the corresponding solution for the empty-space field equations. To obtain the coupled field solutions, we use the method prescribed by Teixeira *et al* (1976, henceforth called the TWS method) to develop the coupled field solutions for electromagnetic and long-range scalar fields from the vacuum solutions. Recently, Som *et al* (1977) showed that the method obtained by Janis *et al* (1969) for developing the coupled field solutions used by Datta and Rao is a special case of the TWS method.

In § 2 we present a brief review of the TWS method. § 3 contains the coupled field equations and their solutions. In the following sections we analyse the singular structure of the metric obtained.

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2. TWS method

In this section we review the TWS method for developing the coupled field solutions for the Einstein–Maxwell-scalar equations.

If the static metric of the line element

$$ds^2 = e^{2V}(dx^0)^2 - e^{-2V}h_{ij}\,dx^i\,dx^j, \tag{2.1}$$

where V and h_{ij} are functions of x_i (italic indices vary from 1 to 3), represents the vacuum solutions of the Einstein equations, then the metric of the line element

$$ds^2 = e^{2\Psi}(dx^0)^2 - e^{-2\Psi}h_{ij}\,dx^i\,dx^j \tag{2.2}$$

is a static solution of the Einstein-Maxwell-scalar equations, where

$$\Psi = -\ln(A\cosh CV + B\sinh CV). \tag{2.3}$$

A, B and C are constants of integration. The scalar field is then given by

$$S = \pm c_p V \tag{2.4}$$

where the upper (lower) sign corresponds to the attractive (repulsive) type of scalar field and c_p is the constant of integration. The constants C and c_p are related by

$$C = (1 \mp c_p^2)^{1/2}. \tag{2.5}$$

The electrostatic potential $\phi(x^i)$ and the electrostatic field $F_{0i}(x^i)$ are

$$\phi(x^i) = -\frac{a}{AC}e^{\Psi}\sinh CV \tag{2.6}$$

$$F_{0i}(x^i) = a\,e^{2\Psi}V_{,i} \tag{2.7}$$

where a is another constant of integration. The four constants A, B, C and a are related by

$$B^2 - A^2 = a^2/C^2. \tag{2.8}$$

One can easily generalise (2.3) by including a magnetostatic field

$$F^{ij} = (b/a)\sqrt{-g}\,\epsilon^{0ijk}\phi_{,k} \tag{2.9}$$

where b is a constant related to the angle of duality rotation θ by $\tan\theta = -b/a$. The relation (2.8) then takes the form

$$B^2 - A^2 = (a^2 + b^2)/C^2. \tag{2.10}$$

In the following sections, we are concerned only with the electrostatic fields, so that A, B, C and a satisfy (2.8).

3. The coupled field equations and their solutions

In the region of space–time containing an electromagnetic field satisfying Maxwell's equations

$$F^{ij}_{;j} = 0 \tag{3.1}$$

$$F_{[ij;\lambda]} = 0 \tag{3.2}$$

where

$$F_{ij} = A_{j,i} - A_{i,j} \tag{3.3}$$

and a long-range scalar field satisfying

$$g^{ij} S_{;ij} = 0 \tag{3.4}$$

the field equations are given by

$$R^{\mu}{}_{\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu} R = -8\pi (E^{\mu}{}_{\nu} + S^{\mu}{}_{\nu}) \tag{3.5}$$

where

$$E^{\mu}{}_{\nu} = \frac{1}{4\pi}(-F_{\nu\alpha}F^{\alpha\mu} + \tfrac{1}{4}\delta^{\mu}{}_{\nu}F_{\alpha\beta}F^{\beta\alpha}) \tag{3.6}$$

$$S^{\mu}{}_{\nu} = \frac{\pm 1}{4\pi}(S^{;\mu}S_{;\nu} - \tfrac{1}{2}\delta^{\mu}{}_{\nu}S^{;\alpha}S_{;\alpha}). \tag{3.7}$$

To obtain the static solution of (3.5) we consider the Curzon metric as empty-space solutions of the Einstein field equations. In Weyl's canonical coordinates, the Curzon metric is given by

$$ds^2 = \exp(-2m/\rho)(dx^0)^2 - \exp(2m/\rho)[\exp(-m^2r^2/\rho^4)(dr^2 + dz^2) + r^2\, d\phi^2] \tag{3.8}$$

where $\rho = (r^2 + z^2)^{1/2}$.

The Weyl canonical coordinates require that the field equations must satisfy (Synge 1976).

$$R^3_3 + R^0_0 = 0. \tag{3.9}$$

Since for an axially symmetric field in Weyl's canonical coordinates, all the field components are functions of r and z, it is evident from (2.7) that the only non-zero components of the electromagnetic field tensor are F_{01} and F_{02}. Then from (3.6) we find that

$$\begin{aligned} E^1_1 &= -E^2_2 = -\frac{1}{8\pi}(F_{01}F^{10} - F_{02}F^{20}) \\ E^3_3 &= -E^0_0 = +\frac{1}{8\pi}(F_{01}F^{10} + F_{02}F^{20}) \end{aligned} \tag{3.10}$$

and from (3.7)

$$S^1_1 = -S^2_2 = \frac{\pm 1}{8\pi}(S^{;1}S_{;1} - S^{;2}S_{;2}) \tag{3.11}$$

so that $S^1_1 + S^2_2 = 0$. This implies that $R = 8\pi(S^3_3 + S^0_0)$. Then from (3.5) we have

$$R^0_0 + R^3_3 - R = -8\pi(S^0_0 + S^3_3)$$

so that $R^0_0 + R^3_3 = 0$. One can now easily obtain the solutions of (3.5) from (3.8) following the presciption given by the TWS method. It leads immediately to

$$\begin{aligned} ds^2 = {} & \{A \cosh(Cm/\rho) + [A^2 + (a^2/C^2)^{1/2} \sinh(Cm/\rho)\}^{-2}(dx^0)^2 \\ & - \{A \cosh(Cm/\rho) + [A^2 + (a^2/C^2)]^{1/2} \sinh(Cm/\rho)\}^2[\exp(-m^2r^2/\rho^4) \\ & \times (dr^2 + dz^2) + r^2\, d\phi^2]. \end{aligned} \tag{3.12}$$

The scalar field (2.4) is then given by

$$S_{,1} = \pm c_{\rm p} mr/\rho^3 \qquad S_{,2} = \pm c_{\rm p} mz/\rho^3. \tag{3.13}$$

The electrostatic potential ϕ and the electrostatic field F_{0i} are

$$\phi = +(a/AC)\,{\rm e}^{\Psi}\sinh(Cm/\rho) \tag{3.14}$$

$$\begin{aligned} F_{01} &= ma\,{\rm e}^{2\Psi} r/\rho^3 \\ F_{02} &= ma\,{\rm e}^{2\Psi} z/\rho^3. \end{aligned} \tag{3.15}$$

The constant a admits an easy interpretation from the asymptotic behaviour of the metric (3.12). As $\rho \to \infty$, $A = 1$, $\phi \to ma/\rho$ and $F_{0i} \to max_i/\rho^3$, where $x_1 = r$ and $x_2 = z$. Then $ma \overset{def}{=} q$, so that (3.14) and (3.15) reduce to

$$\begin{aligned} \phi &= (q/mC)\,{\rm e}^{\Psi}\sinh(Cm/\rho) \\ F_{0i} &= q\,{\rm e}^{2\Psi} x_i/\rho^3. \end{aligned} \tag{3.15$'$}$$

When $a = 0$, the metric (3.12) is equivalent to that obtained by Gautreau (1969). The metric obtained by Datta and Rao (1973) corresponds to the case $A = 0$, $a = C$. If one demands that when $a = 0$, $c_{\rm p} = 0$, then the metric corresponds to (3.8) and one must have $A = 1$. So the metric (3.12) takes the form

$$\begin{aligned} {\rm d}s^2 = &\{\cosh(Cm/\rho) + [1 + (a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^{-2}({\rm d}x^0)^2 \\ &- \{\cosh(Cm/\rho) + [1 + (a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2[\exp(-m^2r^2/\rho^4) \\ &\times ({\rm d}r^2 + {\rm d}z^2) + r^2\,{\rm d}\phi^2]. \end{aligned} \tag{3.16}$$

The metric (when $c_{\rm p} = 0$, i.e. $C = 1$) may be called the Reissner–Nordström equivalent for the Curzon metric. The metric (3.16) reduces to the original vacuum metric (3.8) when $a = 0$. In the following section we consider the metric (3.16).

4. Directional singularity

To analyse the directional properties of the singularity we compute the Kretschmann scalar

$$\begin{aligned} \alpha = R^{\mu\nu\rho\sigma}R_{\mu\nu\rho\sigma} \\ = 4&\{\cosh(Cm/\rho) + [1 + (a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^{-4}\exp(2m^2r^2/\rho^4) \\ &\times [(R^1{}_{212})^2 + (R^3{}_{131})^2 + 2(R^3{}_{132})^2 + (R^3{}_{232})^2 \\ &+ (R^0{}_{101})^2 + 2(R^0{}_{102})^2 + (R^0{}_{202})^2 + (\mathscr{R}^0{}_{303})^2] \end{aligned} \tag{4.1}$$

where

$$\mathscr{R}^0{}_{303} = \frac{\exp(-m^2r^2/\rho^4)}{r^2} R^0{}_{303}.$$

The surviving components of $R^\mu{}_{\nu\rho\sigma}$ are given in the Appendix. When $\rho \to 0^\pm$, a careful examination of all of the eight surviving components shows that $\exp(2m^2r^2/\rho^4)$ in (4.1) dominates and tends to infinity as $\rho \to 0$, indicating an intrinsic singularity along every trajectory except for an approach along the Z axis. As the origin is approached along the Z axis, i.e. $r=0$, we find that the terms

$$\{\cosh(mC/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^{-n} \qquad \text{where } n=2,3,4$$

dominate and tend to zero as $\rho \to 0$, suggesting that it is not the location of an intrinsic singularity. If one considers the space-like geodesic defined by the semi-Z axis, it is infinitely long in the sense of the affine path parameter. So the space-like half-geodesic is complete, indicating that it does not define a singularity in the direction $\rho \to 0$.

5. The equipotential areas

To analyse the structure of the singularities we follow the procedures used by Stachel (1968). A singularity will be considered to be localised if the invariantly defined equipotential area approaches zero as the coordinate location of the singularity is approached. To compute the equipotential area, we re-express (3.12) in the spherical coordinates

$$\begin{aligned} ds^2 = &\{\cosh(mC/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(mC/\rho)\}^{-2}(dx^0)^2 \\ &-\{\cosh(mC/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(mC/\rho)\}^2 \\ &\times[\exp(-m^2\sin^2\theta/\rho^2)(d\rho^2+\rho^2\,d\theta^2)+\rho^2\sin^2\theta\,d\phi^2]. \end{aligned} \tag{5.1}$$

The area of the equipotential surfaces $\rho=\text{constant}$, $x^0=\text{constant}$ has the form

$$\begin{aligned} A = &4\pi\rho^2\{\cosh(mC/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(mC/\rho)\}^2 \\ &\times\int_0^1 \exp(-m^2\sin^2\theta)/2\rho^2)\,d(\cos\theta). \end{aligned} \tag{5.2}$$

A careful examination of (5.2) leads to

$$\begin{aligned} &A(\rho)\to\infty \qquad \rho\to 0^+ \\ &A(\rho)\to\infty \qquad \rho\to\infty^+ \qquad \text{for } m>0, \end{aligned}$$

indicating that $A(\rho)$ has at least one minimum. This implies that the equipotential surfaces shrink in area as ρ decreases from a large value until they reach a minimum. On further decreasing ρ in value, they start to increase in area tending to infinity as $\rho \to 0^+$.

6. Discussion

Starting from the Curzon solution, we obtained the solution (3.16) of the Einstein–Maxwell-scalar field equations. When the scalar field vanishes the solution represents the static charged Curzon metric. It is found that the addition of a long-range scalar field and the electromagnetic field does not produce any change in the properties of quantities defined invariantly such as the Kretschmann scalar and the areas of the equipotential surfaces.

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Appendix

The surviving components of the Riemann–Christoffel tensor with respect to the metric (3.12) are

$$R^{1}{}_{212}=\frac{m}{\rho^3}\left(C^2+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}-\frac{m^2}{\rho^4}\left(1+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)$$

$$R^{3}{}_{131}=\frac{m^2}{\rho^6}\left(C^2z^2+z^2-r^2+\frac{a^2(z^2-r^2)}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)+\left(\frac{m(r^2-2z^2)}{\rho^5}+\frac{m^3(r^4-3z^2r^2)}{\rho^9}\right)\times\left(C^2+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$R^{0}{}_{101}=-\frac{m^2}{\rho^6}\left(C^2(z^2-2r^2)+\frac{a^2(z^2-3r^2)}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)-\left(\frac{m}{\rho^3}+\frac{m(r^2-2z^2)}{\rho^5}+\frac{m^3(r^4-3z^2r^2)}{\rho^9}\right)\times\left(C^2+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$R^{2}{}_{232}=-\frac{m^2}{\rho^6}\left(z^2-r^2-C^2r^2+\frac{a^2(z^2-r^2)}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)-\left(\frac{m}{\rho^3}+\frac{m(r^2-2z^2)}{\rho^5}+\frac{m^3(r^4-3z^2r^2)}{\rho^9}\right)\times\left(C^2+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$R^{0}{}_{202}=\frac{m^2}{\rho^6}\left(C^2(2z^2-r^2)+\frac{a^2(3z^2-r^2)}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)+\left(\frac{m(r^2-2z^2)}{\rho^5}+\frac{m^3(r^4-3z^2r^2)}{\rho^9}\right)\times\left(C^2+\frac{a^2}{\{\cosh(Cm/\rho)+[1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$R^3{}_{132} = -\frac{m^2 rz}{\rho^6}\left(C^2 + 2 + \frac{2a^2}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)$$
$$+\left(\frac{m}{\rho^5}3rz - \frac{m^3(rz^3 - 3zr^3)}{\rho^9}\right)$$
$$\times\left(C^2 + \frac{a^2}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$R^0{}_{102} = \frac{m^2}{\rho^6}\left(3C^2 rz + \frac{a^2 4rz}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)$$
$$+\left(-3rz\frac{m}{\rho^5} + \frac{m^3(rz^3 - 3zr^3)}{\rho^9}\right)$$
$$\times\left(C^2 + \frac{a^2}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)^{1/2}$$

$$\mathscr{R}^0{}_{303} = \frac{\exp(-m^2 r^2/\rho^4)}{r^2} R^0{}_{303} = \frac{-m^2}{\rho^4}$$
$$\times\left(C^2 + \frac{a^2}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]^{1/2}\sinh(Cm/\rho)\}^2}\right)$$
$$+\frac{m}{\rho^3}\left(C^2 + \frac{a^2}{\{\cosh(Cm/\rho) + [1+(a^2/C^2)]\sinh(Cm/\rho)\}^2}\right)^{1/2}.$$

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